Optical solitons in presence of higher-order effects

T. Bui Dinh,[*1] V. Cao Long,[1] B. Nguyen Huy,[2] S. Vu Ngoc[2]

[1] *Institute of Physics, University of Zielona Góra*
[2] *Vinh University, Nghe An, Vietnam*

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Abstract — In this paper, we seek special soliton solutions of so-called Higher-Order Nonlinear Schroedinger Equation, which describes the propagation of light waves in nonlinear optical media. Using a developed Jacobi elliptic function expansion method described in [8], we show that in the case of four-order dispersion these traveling wave solutions do not exist. As a byproduct, we regain the results obtained by other authors [8, 9] for the case when only the dispersion to the third order is taken into account.

1. Introduction

It is well known that a great deal of physical processes involved in a given nonlinear problem may be understood in terms of formation of spatial, temporal or spatiotemporal localized structures or solitons. Being a true high-technology application of these mathematical objects, optical solitons have been the objects of intensive theoretical and experimental studies in the past several decades due to their potential applications in long distance communication and all-optical ultrafast switching devices. Generally speaking, they are special solutions of the so-called Higher-Order Nonlinear Schroedinger Equation (HNLS), which describes the propagation of light waves in nonlinear optical media. As a result, it gives us the input-and-output causal relation for light waves in optical fibers. In literature, there is no systematic way of deriving the propagation equation for light waves in nonlinear media for arbitrary orders of dispersion, diffraction and nonlinearities in an arbitrary medium. In [1] we presented a method of derivation for this propagation equation in a nonlinear differential form. As a special case, we introduced this equation systematically in [2].

The study of the above-mentioned localized waves (solitons) is a difficult task, as nonlinear partial differential equations (PDEs) of a given system are usually not integrable. By investigating the integrability of a nonlinear PDE, one gains crucial insight into the structure of the equation and nature of its solutions. With the exception of some analytical solutions obtained by well-known methods (inverse scattering method, Hirota's method [5], Jacobi elliptic function expansion method [8]), solitary wave solutions have to be determined numerically [6]. One of the most effective methods is the Fourier Split-Step method which is used in studying the dynamics of short-pulse splitting in dispersive nonlinear media [3, 4]. In some cases, we should use analytical and numerical methods simultaneously [7].

In this paper, we seek special soliton solutions of HNLS, namely exact traveling wave solutions ([5], Chapter 7). As it has been emphasized in [8], these solutions are some of the most fundamental objects of study in mathematical physics. Using a developed Jacobi elliptic function expansion method described in [8], we show that in the case of four-order dispersion these traveling wave solutions do not exist. Hence, for ultrashort pulses, it is not necessary to include orders of dispersion which are higher than three to obtain solutions of this type. As a byproduct, we regain the results obtained by other authors [8, 9] for the case when only the dispersion to the third order is taken into account.

Our paper is organized as follows. In Section 2, following [8] there is a short description of the developed Jacobi elliptic function expansion method. In Section 3, by using this method for the HNLS with the four-order dispersion, we show that traveling wave solutions do not exist in this case. We also regain dark and bright solitons obtained by other authors [8, 9]. The last section contains our conclusions.

2. Developed Jacobi elliptic function expansion

We consider a nonlinear partial differential equation in a general form

$$N\left(F,|F|,\frac{\partial F}{\partial t},\frac{\partial F}{\partial x},\frac{\partial^2 F}{\partial t^2},\frac{\partial^2 F}{\partial x^2},\frac{\partial^2 F}{\partial x\partial t}\cdots\right)=0. \tag{2.1}$$

We seek the traveling wave solutions of the form

$$F=u(\xi)e^{i(kx-\omega t)},\ \ \xi=cx-\lambda t+x_0, \tag{2.2}$$

where u(ξ) is a real function, λ is a constant parameter and *k* and ω denote the wave number and frequency, respectively. Substituting (2.2) into (2.1) we obtain an ordinary differential equation

$$N\left(u,\frac{du}{d\xi},\frac{d^2u}{d\xi^2},\frac{d^3u}{d\xi^3},\cdots\right)=0. \tag{2.3}$$

We take the *ansatz* of the solution in the form of a finite series of Jacobi elliptic functions *cn(ξ,m)*(or *sn(ξ,m)*), i.e.

$$u(\xi)=\sum_{j=0}^{n}a_j cn^j(\xi). \tag{2.4}$$

in which a_j are constants which will be determined later and the highest degree of the function *u* is

$$O(u(\xi)) = n. \tag{2.5}$$

It follows from the properties of Jacobi elliptic functions that the highest degree of derivatives is taken as

$$O\left(d^p u(\xi)/d\xi^p\right) = n + p. \tag{2.6}$$

n in (2.4) is selected in such a way that the highest degree of derivatives is equal to the degree of the nonlinear term. Substituting (2.4) into (2.3) and equating the coefficients of all power of cn(ξ). sn(ξ), dn(ξ) to zero leads to a set of algebraic equations for a_j. By solving these equations, we obtain the final result for u in the form (2.4).

3. The higher-order propagation equation

As it has been emphasized in Sec. 1, a systematic derivation for light propagation equation in a nonlinear medium is given in [1, 2]. In the case of ultrashort light pulses (femtosecond pulses which have much potential for future technology), in comparison with the nonlinear Schroedinger equation (NLS), higher-order terms should be taken into account. For this reason, we consider the modified NLS equation in the form

$$\begin{aligned}\frac{\partial E}{\partial z} = i\left(\alpha_1 \frac{\partial^2 E}{\partial t^2} + \alpha_2 \frac{\partial^4 E}{\partial t^4} + \alpha_3 |E|^2 E\right) + \alpha_4 \frac{\partial^3 E}{\partial t^3} + \\ + \alpha_5 \frac{\partial\left(|E|^2 E\right)}{\partial t} + \alpha_6 E \frac{\partial |E|^2}{\partial t}\end{aligned}, \tag{3.1}$$

where the real parameters α_i (i=1,...,6) have the following physical interpretations: $\alpha_1$corresponds to the group velocity dispersion (GVD), α_2 to the four-order dispersion (FOD), α_3 to self-phase modulation (SPM), α_4 to third-order dispersion (TOD), α_5 to self-steepening (SS) and α_6 to the self frequency shift (SFS) arising from stimulated Raman scattering (SRS). In order to find traveling wave solutions of Eq.(3.1), we use the developed Jacobi elliptic function expansion method described above. Firstly, we write the electric field in the form

$$E(z,t) = u(\xi)\exp[i.(kz - \omega.t)],\ \xi = ct - \lambda z + z_0. \tag{3.2}$$

Substituting (3.2) into (3.1) we obtain

$$\begin{aligned}-\lambda u' + iku = i\alpha_2\left[c^4 u'''' + \omega^4 u - 6c^2\omega^2 u'' + 4i\left(c\omega^3 u' - c^3\omega u'''\right)\right] + \\ + i\alpha_1\left(c^2 u'' - 2ic\omega u' - \omega^2 u\right) i\alpha_3 u^3 + + \alpha_5\left(3cu^2 u' - i\omega u^3\right) + \\ + \alpha_4\left[c^3 u''' - 3c\omega^2 u' + i\left(\omega^3 u - 3c^2\omega u''\right)\right] + 2\alpha_6 c u^2 u'.\end{aligned} \tag{3.3}$$

Separating the real and imaginary parts of this equation leads to the following system of equations

$$\begin{aligned}c^3\left(\alpha_4 + 4\alpha_2\omega\right)u''' + \left(\lambda - 3\alpha_4 c\omega^2 + 2\alpha_1 c\omega - 4\alpha_2 c\omega^3\right)u' + \\ + \left(3\alpha_5 + 2\alpha_6\right)cu^2 u' = 0,\end{aligned} \tag{3.4a}$$

$$\begin{aligned}c^4\alpha_2 u'''' + \left(c^2\alpha_1 - 6\alpha_2 c^2\omega^2 - 3c^2\omega\alpha_4\right)u'' + \\ + \left(-k - \alpha_1\omega^2 + \alpha_2\omega^4 + \alpha_4\omega^3\right)u + \left(\alpha_3 - \alpha_5\omega\right)u^3 = 0.\end{aligned} \tag{3.4b}$$

Now using developed Jacobi elliptic function expansion method described in Sec. 2 we calculate the highest degree of derivative $O(d^4u(\xi)/d\xi^4)=n+4$ and the degree of the nonlinear term $O(u^3(\xi))=3n$. Equating these numbers leads to $n = 2$. It follows that the function $u(\xi)$ can be obtained from the form

$$u(\xi) = a_0 + a_1 cn(\xi) + a_2 cn^2(\xi). \tag{3.5}$$

For the sake of simplicity, we suppose that $a_0=a_1=0$, then $u(\xi)=a_2 cn^2(\xi)$. Substituting this expression into Eq. (3.4b) gives

$$A_6 cn^6(\xi) + A_4 cn^4(\xi) + A_2 cn^2(\xi) + A_0 = 0, \tag{3.6}$$

where coefficients A_i contain different parameters involved in the problem. Equating the coefficient of the first term in (3.6) to zero leads to $A_6 = 48\alpha_2 c^4 a_2 m^4 + 24\alpha_2 c^4 a_2 m^2 = 0$. Because a_2, m, c should be different from zero, we have $\alpha_2=0$. This means that if the term FOD is taken into account, the traveling wave solutions do not exist. We conclude that for the existence of solutions in this type, the orders of dispersion higher than three should not be taken into account. Then we can rewrite (3.4a) and (3.4b) in the form (with $\alpha_2=0$)

$$c^3\alpha_4 u''' + \left(\lambda - 3\alpha_4 c\omega^2 + 2\alpha_1 c\omega\right)u' + \left(3\alpha_5 + 2\alpha_6\right)cu^2 u' = 0 \tag{3.7a}$$

$$\left(c^2\alpha_1 - 3c^2\omega\alpha_4\right)u'' + \left(-k - \alpha_1\omega^2 + \alpha_4\omega^3\right)u + \left(\alpha_3 - \alpha_5\omega\right)u^3 = 0 \tag{3.7b}$$

Differentiating two sides of Eq. (3.7b) with respect to the ξ gives us

$$u''' + \frac{\left(-k - \alpha_1\omega^2 + \alpha_4\omega^3\right)}{\left(c^2\alpha_1 - 3c^2\omega\alpha_4\right)}u' + 3\frac{\left(\alpha_3 - \alpha_5\omega\right)}{\left(c^2\alpha_1 - 3c^2\omega\alpha_4\right)}u^2 u' = 0. \tag{3.8}$$

Comparing (3.8) with (3.7a) leads to formulas for ω and k:

$$\omega = \left[\alpha_1\left(3\alpha_5 + 2\alpha_6\right) - 3\alpha_3\alpha_4\right]\left[6\alpha_4\left(\alpha_5 + \alpha_6\right)\right]^{-1}, \tag{3.9}$$

$$k = -\frac{1}{c\alpha_4}\left(\lambda - 3\alpha_4 c\omega^2 + \alpha_1 c\omega\right) - \alpha_1\omega^2 + \omega^3\alpha_4. \tag{3.10}$$

Then Eqs. (3.7a), (3.7b) reduce to

$$u'' + Au + Bu^3 = 0, \tag{3.11}$$

where

$$A = \frac{2\alpha_1\omega + \lambda - 3\alpha_4\omega^2}{c^2\alpha_4};\quad B = \frac{3\alpha_5 + 2\alpha_6}{3c^2\alpha_4}. \tag{3.12}$$

Now, we use the formalism described in Sec.2 for the Eq.(3.11). Firstly, we calculate $O(d^2u/d\xi^2)=n+2$ and $O(u^3(\xi))=3n$. Then n=1 and we can write u(ξ) in the following form:

$$u(\xi) = a_0 + a_1 cn(\xi). \tag{3.13}$$

Substituting (3.13) into (3.11) and equating the coefficients of all powers of cn(ξ) to zero yields the values of unknown parameters a_0, a_1, c, λ. We have performed this step by MAPLE and obtained:

$$a_0 = 0, a_1 = \sqrt{\frac{6\alpha_4}{2\alpha_6 + 3\alpha_5}}mc,\ \lambda = -2m^2c^2\alpha_4 - 2\alpha_1\omega + c^2\alpha_4 + 3\alpha_4\omega^2, \tag{3.14}$$

while c is an arbitrary constant and m is the modulus number of the Jacobi elliptic functions. Then the traveling wave solutions of the propagation equation (3.1) have the following form

$$E(z,t)=\sqrt{\frac{6\alpha_4}{2\alpha_6+3\alpha_5}}mc.\exp[i.(kz-\omega t)].$$
$$.cn\left[ct-\left(-2m^2c^2\alpha_4-2\alpha_1\omega+c^2\alpha_4+3\alpha_4\omega^2\right)z+z_0\right] \quad (3.15)$$

where the expressions for k, ω are given by (3.9) and (3.10). When m tends to 1, we obtain a bright soliton solution

$$E(z,t)=\sqrt{\frac{6\alpha_4}{2\alpha_6+3\alpha_5}}c.\ \exp[i.(kz-\omega t)].$$
$$.\sec h\left[ct-\left(-c^2\alpha_4-2\alpha_1\omega+3\alpha_4\omega^2\right)z+z_0\right]. \quad (3.16)$$

Now instead of (3.13) we use the *ansatz*

$$u(\xi)=a_0+a_1sn(\xi)\ . \quad (3.17)$$

By substituting (3.17) into (3.11) we obtain $a_0=0,\ a_1=\sqrt{\frac{-6\alpha_4}{2\alpha_6+3\alpha_5}}mc, \lambda=c^2\alpha_4-2\alpha_1\omega+m^2c^2\alpha_4+3\alpha_4\omega^2$, (3.18)

while c is also an arbitrary constant. Then the solution of Eq.(3.1) has the following form:

$$E(z,t)=\sqrt{\frac{-6\alpha_4}{2\alpha_6+3\alpha_5}}mc.\exp[i.(kz-\omega t)].$$
$$.sn\left[ct-\left(c^2\alpha_4-2\alpha_1\omega+m^2c^2\alpha_4+3\alpha_4\omega^2\right)z+z_0\right], \quad (3.19)$$

where the expressions for k, ω are given by (3.9) and (3.10). When the modulus number m tends to 1 we have a dark soliton solution in the following form:

$$E(z,t)=\sqrt{\frac{-6\alpha_4}{2\alpha_6+3\alpha_5}}c.\tanh\left[ct-\left(2c^2\alpha_4-2\alpha_1\omega+3\alpha_4\omega^2\right)z+z_0\right].$$
$$.\exp[i.(kz-\omega t)] \quad (3.20)$$

Our expressions (3.16) and (3.20) are just the results previously obtained by several authors (e.g. the formulas (56), (58) in [8] and (9), (12) in [9]). We demonstrate these for some values of parameters involved in Figures 1-4.

4. Conclusions

In our paper we used a developed Jacoby elliptic function expansion method to find traveling wave solutions for HNLS which describes light propagation in fibers. We showed that we cannot have solutions of this type when we include higher orders of dispersion. As a byproduct, we obtained in a simple way bright and dark soliton solutions discovered before by other authors.

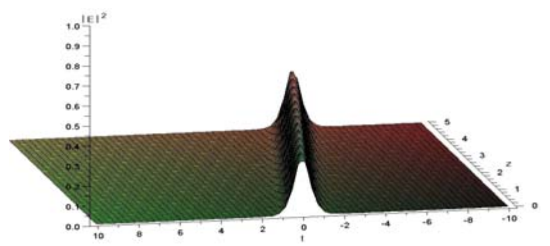

Fig. 1: Bright soliton (3.16) with c =2, α_1= - 0.5, α_3 =1, α_4=1/24, α_5=0.8, α_6=0.5 and z_0=0.

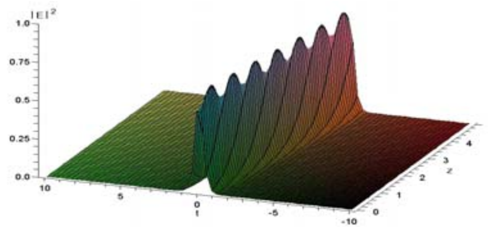

Fig.2. Bright soliton (3.16) with c =3, α_1= - 0.5, α_3 =1, α_4=1/24, α_5=0.8, α_6=0.5 and z_0=0.

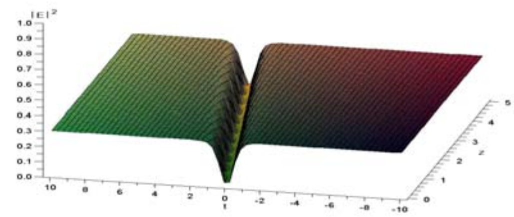

Fig.3. Dark soliton (3.19) with c =2, α_1= -0.5, α_3 =1, α_4=-1/24, α_5=0.8, α_6=0.5 and z_0=0.

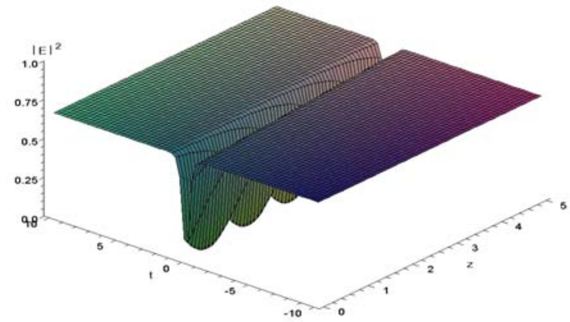

Fig.4. Dark soliton (3.19) with c =3, α_1= 0.5, α_3 =1, α_4=-1/24, α_5=0.8, α_6=0.5 and z_0=0.

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